# SYMPLECTIC $U_7, U_8$ AND $U_9$ SINGULARITIES

ŻANETA TRĘBSKA

Abstract. We use the method of algebraic restrictions to classify symplectic $U_7, U_8$ and $U_9$ singularities. We use discrete symplectic invariants to distinguish symplectic singularities of the curves. We also give the geometric description of symplectic classes.

## 1. Introduction

In this paper we examine the singularities which are in the list of the simple 1-dimensional isolated complete intersection singularities in the space of dimension greater than 2, obtained by Giusti ([G], [AVG]). Isolated complete intersection singularities (ICIS) were intensively studied by many authors (e. g. see [L]), because of their interesting geometric, topological and algebraic properties. Here using the method of algebraic restrictions we obtain the complete symplectic classification of the singularities of type $U_7, U_8$ and $U_9$. We calculate discrete symplectic invariants for symplectic orbits of the curves and we give their geometric description. It allows us to explore the specific singular nature of these classical singularities that only appears in the presence of the symplectic structure.

We study the symplectic classification of singular curves under the following equivalence:

**Definition 1.1.** Let $N_1, N_2$ be germs of subsets of symplectic space $(\mathbb{R}^{2n}, \omega)$. $N_1, N_2$ are *symplectically equivalent* if there exists a symplectomorphism-germ $\Phi : (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega)$ such that $\Phi(N_1) = N_2$.

We recall that $\omega$ is a symplectic form if $\omega$ is a smooth nondegenerate closed 2-form, and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if $\Phi$ is diffeomorphism and $\Phi^*\omega = \omega$.

Symplectic classification of curves was initiated by V. I. Arnold. In [A1] and [A2] the author studied singular curves in symplectic and contact spaces and introduced the local symplectic and contact algebra. He discovered new symplectic invariants of singular curves. He proved that the $A_{2k}$ singularity of a planar curve (the orbit with respect to standard $\mathcal{A}$-equivalence of parameterized curves) split into exactly $2k+1$ symplectic singularities (orbits with respect to symplectic equivalence of parameterized curves). He distinguished different symplectic singularities by different orders of tangency of the parameterized curve to the *nearest* smooth Lagrangian submanifold. Arnold posed a problem of expressing these invariants in terms of the local algebra's interaction with the symplectic structure and he proposed calling this interaction the local symplectic algebra.

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In [IJ1] G. Ishikawa and S. Janeczko classified symplectic singularities of curves in the 2-dimensional symplectic space. All simple curves in this classification are quasi-homogeneous.

We recall that a subset $N$ of $\mathbb{R}^m$ is *quasi-homogeneous* if there exist a coordinate system $(x_1, \cdots, x_m)$ on $\mathbb{R}^m$ and positive numbers $w_1, \cdots, w_m$ (called weights) such that for any point $(y_1, \cdots, y_m) \in \mathbb{R}^m$ and any $t > 0$ if $(y_1, \cdots, y_m)$ belongs to $N$ then the point $(t^{w_1} y_1, \cdots, t^{w_m} y_m)$ belongs to $N$.

The generalization of results in [IJ1] to volume-preserving classification of singular varieties and maps in arbitrary dimensions was obtained in [DR]. A symplectic form on a 2-dimensional manifold is a special case of a volume form on a smooth manifold.

The stably simple symplectic singularities of parameterized curves (in the $\mathbb{C}$-analytic category) were studied by P. A. Kolgushkin in [K].

In [Z] was developed the local contact algebra. The main results were based on the notion of the algebraic restriction of a contact structure to a subset $N$ of a contact manifold.

In [DJZ2] new symplectic invariants of singular quasi-homogeneous subsets of a symplectic space were explained by the algebraic restrictions of the symplectic form to these subsets.

The algebraic restriction is an equivalence class of the following relation on the space of differential $k$-forms:

Differential $k$-forms $\omega_1$ and $\omega_2$ have the same *algebraic restriction* to a subset $N$ if $\omega_1 - \omega_2 = \alpha + d\beta$, where $\alpha$ is a $k$-form vanishing on $N$ and $\beta$ is a $(k-1)$-form vanishing on $N$.

In [DJZ2] the generalization of Darboux-Givental theorem ([AG]) to germs of arbitrary subsets of the symplectic space was obtained. This result reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. For non-quasi-homogeneous subsets there is one more cohomological invariant apart from the algebraic restriction ([DJZ2], [DJZ1]). The dimension of the space of algebraic restrictions of closed 2-forms to a 1-dimensional quasi-homogeneous isolated complete intersection singularity $C$ is equal to the multiplicity of $C$ ([DJZ2]). In [D1] it was proved that the space of algebraic restrictions of closed 2-forms to a 1-dimensional (singular) analytic variety is finite-dimensional. In [DJZ2] the method of algebraic restrictions was applied to various classification problems in a symplectic space. In particular the complete symplectic classification of classical $A$-$D$-$E$ singularities of planar curves and $S_5$ singularity was obtained. Most of different symplectic singularity classes were distinguished by new discrete symplectic invariants: the index of isotropy and the symplectic multiplicity.

In [DT1] following ideas from [A1] and [D1] new discrete symplectic invariants - the Lagrangian tangency orders were introduced and used to distinguish symplectic singularities of simple planar curves of type $A$-$D$-$E$, symplectic $T_7$ and $T_8$ singularities.

The complete symplectic classification of the isolated complete intersection singularities $S_\mu$ for $\mu > 5$ and $W_8$, $W_9$ singularities were obtained in [DT2] and [T] respectively.

The method of algebraic restrictions was successfully used by W. Domitrz in [D2] to classify the 0-dimensional ICIS (multiple points) in a symplectic space.

In this paper we obtain the detailed symplectic classification of the $U_7, U_8$ and the $U_9$ singularities. The paper is organized as follows. In Section 2 we recall discrete symplectic invariants (the symplectic multiplicity, the index of isotropy and the Lagrangian tangency orders). Symplectic classification of the $U_7, U_8$ and the $U_9$ singularity is presented in Sections 3, 4 and 5 respectively. The symplectic sub-orbits of this singularities are listed in Theorems 3.1, 4.1 and 5.1. Discrete symplectic invariants for the symplectic classes are calculated in Theorems 3.2, 4.2 and 5.2. The geometric descriptions of the symplectic orbits are presented in Theorems 4.3, 4.3 and 5.3. In Section 6 we recall the method of algebraic restrictions and use it to classify symplectic singularities.

## 2. Discrete symplectic invariants

We can use discrete symplectic invariants to characterize symplectic singularity classes.

The first invariant is a symplectic multiplicity ([DJZ2]) introduced in [IJ1] as a symplectic defect of a curve.

Let $N$ be a germ of a subvariety of $(\mathbb{R}^{2n}, \omega)$.

**Definition 2.1.** The *symplectic multiplicity*, $\mu^{sym}(N)$ of $N$ is the codimension of the symplectic orbit of $N$ in the orbit of $N$ with respect to the action of the group of local diffeomorphisms.

The second invariant is the index of isotropy [DJZ2].

**Definition 2.2.** The *index of isotropy*, $ind(N)$ of $N$ is the maximal order of vanishing of the 2-forms $\omega|_{TM}$ over all smooth submanifolds $M$ containing $N$.

This invariant has geometrical interpretation. An equivalent definition is as follows: the index of isotropy of $N$ is the maximal order of tangency between non-singular submanifolds containing $N$ and non-singular isotropic submanifolds of the same dimension. The index of isotropy is equal to 0 if $N$ is not contained in any non-singular submanifold which is tangent to some isotropic submanifold of the same dimension. If $N$ is contained in a non-singular Lagrangian submanifold then the index of isotropy is $\infty$.

The symplectic multiplicity and the index of isotropy can be described in terms of algebraic restrictions (Propositions 6.4 and 6.5 in Section 6).

There is one more discrete symplectic invariant, introduced in [D1] (following ideas from [A2]) which is defined specifically for a parameterized curve. This is the maximal tangency order of a curve $f : \mathbb{R} \to M$ to a smooth Lagrangian submanifold. If $H_1 = ... = H_n = 0$ define a smooth submanifold $L$ in the symplectic space then the tangency order of a curve $f : \mathbb{R} \to M$ to $L$ is the minimum of the orders of vanishing at 0 of functions $H_1 \circ f, \cdots, H_n \circ f$. We denote the tangency order of $f$ to $L$ by $t(f, L)$.

**Definition 2.3.** The *Lagrangian tangency order* $Lt(f)$ *of a curve* $f$ is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds $L$ of the symplectic space.

The Lagrangian tangency order of the quasi-homogeneous curve in a symplectic space can also be expressed in terms of algebraic restrictions (Proposition 6.6 in Section 6).

In [DT1] the above invariant was generalized for germs of curves and multi-germs of curves which may be parameterized analytically since the Lagrangian tangency order is the same for every 'good' analytic parametrization of a curve.

Consider a multi-germ $(f_i)_{i\in\{1,\cdots,r\}}$ of analytically parameterized curves $f_i$. We have $r$-tuples $(t(f_1,L),\cdots,t(f_r,L))$ for any smooth submanifold $L$ in the symplectic space.

**Definition 2.4.** For any $I\subseteq\{1,\cdots,r\}$ we define *the tangency order of the multi-germ* $(f_i)_{i\in I}$ to $L$:

$$t[(f_i)_{i\in\ I},L]=\min_{i\in\ I} t(f_i,L).$$

**Definition 2.5.** The *Lagrangian tangency order* $Lt((f_i)_{i\in\ I})$ *of a multi-germ* $(f_i)_{i\in I}$ is the maximum of $t[(f_i)_{i\in\ I},L]$ over all smooth Lagrangian submanifolds $L$ of the symplectic space.

## 3. Symplectic $U_7$-singularities

Denote by $(U_7)$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n},\omega)$ which are diffeomorphic to

$$U_7=\{x\in\mathbb{R}^{2n\geq 4}\ : x_1^2+x_2x_3=x_1x_2+x_3^3=x_{\geq 4}=0\}. \tag{3.1}$$

This is the simple 1-dimensional isolated complete intersection singularity $U_7$ ([G], [AVG]). Here $N$ is quasi-homogeneous with weights $w(x_1)=4$, $w(x_2)=5$, $w(x_3)=3$.

We used the method of algebraic restrictions to obtain the complete classification of symplectic singularities of $(U_7)$ presented in the following theorem.

**Theorem 3.1.** *Any submanifold of the symplectic space* $(\mathbb{R}^{2n},\sum_{i=1}^n dp_i\wedge dq_i)$ *where* $n\geq 3$ *(respectively* $n=2$*) which is diffeomorphic to* $U_7$ *is symplectically equivalent to one and only one of the normal forms* $U_7^i, i=0,1,\cdots,7$ *(respectively* $i=0,1,2$*) listed below. The parameters* $c,c_1,c_2$ *of the normal forms are moduli:*

$U_7^0$: $p_1^2+p_2q_1=0,\ \ p_1p_2+q_1^3=0,\ \ q_2=c_1q_1+c_2p_1,\ \ p_{\geq 3}=q_{\geq 3}=0$;
$U_7^1$: $p_2^2\pm p_1q_1=0,\ \ p_1p_2\pm q_1^3=0,\ \ q_2=c_1p_1+\frac{c_2}{2}q_1^2,\ \ p_{\geq 3}=q_{\geq 3}=0$;
$U_7^2$: $p_1^2+q_1q_2=0,\ \ p_1q_1+q_2^3=0,\ \ p_2=c_1p_1q_2+\frac{c_2}{2}p_1^2,\ \ p_{\geq 3}=q_{\geq 3}=0$;
$U_7^3$: $p_1^2+p_2p_3=0,\ p_1p_2+p_3^3=0,\ q_1=cp_1p_3, q_2=0, q_3=\pm p_1p_3,\ p_{\geq 4}=q_{\geq 4}=0$;
$U_7^4$: $p_1^2+p_2p_3=0,\ p_1p_2+p_3^3=0,\ q_1=\frac{c}{3}p_3^3, q_2=0, q_3=-\frac{1}{2}p_1^2,\ p_{\geq 4}=q_{\geq 4}=0$;
$U_7^5$: $p_1^2+p_2p_3=0,\ p_1p_2+p_3^3=0,\ q_1=-\frac{c}{2}p_1p_3^2, q_2=0, q_3=-p_1p_3^2,\ p_{\geq 4}=q_{\geq 4}=0$;
$U_7^6$: $p_1^2+p_2p_3=0,\ p_1p_2+p_3^3=0,\ q_1=0, q_2=0, q_3=\mp\frac{1}{2}p_1^2p_3,\ p_{\geq 4}=q_{\geq 4}=0$;
$U_7^7$: $p_1^2+p_2p_3=0,\ p_1p_2+p_3^3=0,\ q_{\geq 1}=p_{\geq 4}=0$.

3.1. **Distinguishing symplectic classes of $U_7$ by the Lagrangian tangency orders.** A curve $N\in(U_7)$ may be described as a union of two parametrical branches $B_1$ and $B_2$. The branch $B_1$ is smooth so it is contained in some Lagrangian submanifold and thus $Lt(B_1)=\infty$. The branch $B_2$ is singular. The parametrizations of branches are given in Table 1. To characterize the symplectic classes we use the following invariants:

- $Lt = Lt(B_1, B_2) = \max_{\mathcal{L}}(\min\{t(B_1, \mathcal{L}), t(B_2, \mathcal{L})\})$,
- $L_2 = Lt(B_2) = \max_{\mathcal{L}} t(B_2, \mathcal{L})$,

Here $L$ is a smooth Lagrangian submanifold of the symplectic space. We also compute $ind$ (the index of isotropy of $N$) and $ind_2$ (the index of isotropy of the singular component).

**Theorem 3.2.** *Any stratified submanifold $N \in (U_7)$ of a symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 1. The indices of isotropy and the Lagrangian tangency orders of the curve $N$ are presented in the third, fourth, fifth and sixth column of Table 1.*

| class | parametrization of branches of $N$ | | $ind$ | $ind_2$ | $Lt$ | $L_2$ |
|---|---|---|---|---|---|---|
| $(U_7)^0$ | $B_1 : (0, 0, t, 0, 0, \cdots)$ | if $c_1 \neq 0$ | 0 | 0 | 3 | 4 |
| $2n \geq 4$ | $B_2 : (t^4, -t^3, t^5, -c_1 t^3 - c_2 t^4, 0, \cdots)$ | if $c_1 = 0$ | 0 | 0 | 4 | 4 |
| $(U_7)^1$ | $B_1 : (t, 0, 0, c_1 t, , 0, \cdots)$ | | 0 | 0 | 3 | 5 |
| $2n \geq 4$ | $B_2 : (t^5, \mp t^3, t^4, c_1 t^5 + \frac{c_2}{2} t^6, 0, \cdots)$ | | | | | |
| $(U_7)^2$ | $B_1 : (0, t, 0, 0, 0, \cdots)$ | | 0 | 0 | 4 | 5 |
| $2n \geq 4$ | $B_2 : (t^4, t^5, -c_1 t^7 + \frac{c_2}{2} t^8, -t^3, 0, \cdots)$ | | | | | |
| $(U_7)^3$ | $B_1 : (0, 0, t, 0, 0, 0, 0, \cdots)$ | | 1 | 1 | 7 | 7 |
| $2n \geq 6$ | $B_2 : (t^4, -ct^7, t^5, 0, -t^3, \pm t^7, 0, \cdots)$ | | | | | |
| $(U_7)^4$ | $B_1 : (0, 0, t, 0, 0, 0, 0, \cdots)$ | | 1 | 1 | 8 | 8 |
| $2n \geq 6$ | $B_2 : (t^4, -\frac{c}{3} t^9, t^5, 0, -t^3, -\frac{1}{2} t^8, 0, \cdots)$ | | | | | |
| $(U_7)^5$ | $B_1 : (0, 0, t, 0, 0, 0, 0, \cdots)$ | | 2 | $\infty$ | 10 | $\infty$ |
| $2n \geq 6$ | $B_2 : (t^4, -\frac{c}{2} t^{10}, t^5, 0, -t^3, -t^{10}, 0, \cdots)$ | | | | | |
| $(U_7)^6$ | $B_1 : (0, 0, t, 0, 0, 0, 0, \cdots)$ | | 2 | $\infty$ | 11 | $\infty$ |
| $2n \geq 6$ | $B_2 : (t^4, 0, t^5, 0, -t^3, \pm \frac{1}{2} t^{11}, 0, \cdots)$ | | | | | |
| $(U_7)^7$ | $B_1 : (0, 0, t, 0, 0, 0, 0, \cdots)$ | | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_2 : (t^4, 0, t^5, 0, -t^3, 0, 0, \cdots)$ | | | | | |

TABLE 1. The symplectic invariants for symplectic classes of $U_7$ singularity.

**Remark.** The comparison of invariants presented in Table 1 shows that the Lagrangian tangency orders distinguish more symplectic classes than the respective indices of isotropy.

The most of invariants can be calculated by knowing algebraic restrictions for the symplectic classes. We use Proposition 6.5 to calculate the indices of isotropy. $L_2$ is calculated by using Proposition 6.6 for the singular branch. $Lt$ is computed by applying directly the definition of the Lagrangian tangency order and finding a Lagrangian submanifold the nearest to the curve $N$.

3.2. **Identifying the classes $(U_7)^i$ by geometric conditions.**
We can characterize the symplectic classes $(U_7)^i$ by geometric conditions independent of any local coordinate system.

Let $N \in (U_7)$. Denote by $W$ the tangent space at 0 to some non-singular 3-manifold containing $N$. We can define the following subspaces of this space:

$\ell_1$ – the tangent line at 0 to the nonsingular branch $B_1$,

$\ell_2$ – the tangent line at 0 to the singular branch $B_2$

$V$ – the 2-space tangent at 0 to the singular branch $B_2$.

For $N = U_7 =$(3.1) it is easy to calculate that $W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, and $\ell_1 = \mathrm{span}(\partial/\partial x_2)$, $\ell_2 = \mathrm{span}(\partial/\partial x_3)$, $V = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_3)$.

The classes $(U_7)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where $\omega$ is the symplectic form.

**Theorem 3.3.** *Any stratified submanifold $N \in (U_7)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(U_7)^i$ if and only if the couple $(N, \omega)$ satisfies the corresponding conditions in the last column of Table 2.*

| class | normal form | geometric conditions |
|---|---|---|
| $(U_7)^0$ | $[U_7]^0_0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_7},\ c_1 \neq 0$ | $\omega\|_V \neq 0$ and $\omega\|_{\ell_1+\ell_2} \neq 0$ |
| | $[U_7]^0_1 : [\theta_1 + c_2\theta_3]_{U_7}$ | $\omega\|_V \neq 0$ and $\omega\|_{\ell_1+\ell_2} = 0$ |
| $(U_7)^1$ | $[U_7]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{U_7}$ | $\omega\|_V = 0$ but $\ker\omega \neq \ell_2$ |
| $(U_7)^2$ | $[U_7]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{U_7}$ | $\omega\|_V = 0$ and $\ker\omega = \ell_2$ |
| | | $\omega\|_W = 0$ |
| $(U_7)^3$ | $[U_7]^3 : [\pm\theta_4 + c\theta_5]_{U_7}$ | $Lt = L_2 = 7$ |
| $(U_7)^4$ | $[U_7]^4 : [\theta_5 + c\theta_6]_{U_7}$ | $Lt = L_2 = 8$ |
| $(U_7)^5$ | $[U_7]^5 : [\theta_6 + c\theta_7]_{U_7}$ | $Lt = 10,\ L_2 = \infty$ |
| $(U_7)^6$ | $[U_7]^6 : [\pm\theta_7]_{U_7}$ | $Lt = 11,\ L_2 = \infty$ |
| $(U_7)^7$ | $[U_7]^7 : [0]_{U_7}$ | $N$ is contained in a smooth Lagrangian submanifold |

Table 2. Geometric interpretation of singularity classes of $U_7$. ($W$ is the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n\geq 4}, \omega)$ containing $N \in (U_7)$. The forms $\theta_1, \ldots, \theta_7$ are described in Theorem 6.8 on the page 14.)

*Sketch of the proof of Theorem 3.3.* We have to show that the conditions in the row of $(U_7)^i$ are satisfied for any $N \in (U_7)^i$. Each of the conditions in the last column of Table 2 is invariant with respect to the action of the group of diffeomorphisms in the space of pairs $(N, \omega)$. Because each of these conditions depends only on the algebraic restriction $[\omega]_N$ we can take the simplest 2-forms $\omega^i$ representing the normal forms $[U_7]^i$ for algebraic restrictions and we can check that the pair $(U_7, \omega = \omega^i)$ satisfies the condition in the last column of Table 2. By simple calculation and observation of the Lagrangian tangency orders we obtain that the conditions corresponding to the classes $(U_7)^i$ are satisfied. $\square$

## 4. Symplectic $U_8$-singularities

Denote by $(U_8)$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$U_8 = \{x \in \mathbb{R}^{2n \geq 4} : x_1^2 + x_2 x_3 = x_1 x_2 + x_1 x_3^2 = x_{\geq 4} = 0\}. \tag{4.1}$$

This is the simple 1-dimensional isolated complete intersection singularity $U_8$ ([G], [AVG]). Here $N$ is quasi-homogeneous with weights $w(x_1) = 3$, $w(x_2) = 4$, $w(x_3) = 2$.

We used the method of algebraic restrictions to obtain the complete classification of symplectic singularities of $(U_8)$ presented in the following theorem.

**Theorem 4.1.** *Any submanifold of the symplectic space $(\mathbb{R}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)$ where $n \geq 3$ (respectively $n = 2$) which is diffeomorphic to $U_8$ is symplectically equivalent to one and only one of the normal forms $U_8^i, i = 0, 1, \cdots, 8$ listed below. The parameters $c, c_1, c_2$ of the normal forms are moduli:*

$U_8^0$: $p_1^2 + p_2 q_1 = 0, \quad p_1 p_2 + p_1 q_1^3 = 0, \quad q_2 = c_1 q_1 - c_2 p_1, \quad p_{\geq 3} = q_{\geq 3} = 0;$
$U_8^1$: $p_1^2 \pm p_2 q_2 = 0, \quad p_1 p_2 + p_1 q_2^2 = 0, \quad q_1 = c_1 p_2 + \frac{c_2}{2} q_2^2, \quad p_{\geq 3} = q_{\geq 3} = 0;$
$U_8^2$: $p_1^2 + q_1 q_2 = 0, \quad p_1 q_1 + p_1 q_2^2 = 0, \quad p_2 = c_1 p_1 q_2 + \frac{c_2}{2} p_1^2, \quad p_{\geq 3} = q_{\geq 3} = 0;$
${U_8}_5^{3,0}$: $p_1^2 + q_1 q_2 = 0, \quad p_1 q_1 + p_1 q_2^2 = 0, \quad p_2 = -\frac{1}{3} p_1 q_2 + \frac{c_1}{2} p_1^2 + c_2 p_1 q_2^2, \quad p_{\geq 3} = q_{\geq 3} = 0;$
${U_8}_\infty^{3,0}$: $p_1^2 + q_1 q_2 = 0, \quad p_1 q_1 + p_1 q_2^2 = 0, \quad p_2 = 2 p_1 q_2 + \frac{c_1}{2} p_1^2 + \frac{c_2}{2} p_1^2 q_2, \quad p_{\geq 3} = q_{\geq 3} = 0;$
${U_8}^{3,1}$: $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_1 p_3^2 = 0,\ q_1 = q_2 = 0, q_3 = -p_1 p_3 - \frac{c}{2} p_1^2,\ p_{>3} = q_{>3} = 0;$
$U_8^4$: $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_1 p_3^2 = 0,\ q_1 = q_2 = 0, q_3 = \mp \frac{1}{2} p_1^2 - c p_1 p_3^2,\ p_{>3} = q_{>3} = 0;$
$U_8^5$: $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_1 p_3^2 = 0,\ q_1 = q_2 = 0, q_3 = -p_1 p_3^2 - \frac{c}{2} p_1^2 p_3,\ p_{>3} = q_{>3} = 0;$
$U_8^6$: $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_1 p_3^2 = 0,\ q_1 = q_2 = 0, q_3 = \mp \frac{1}{2} p_1^2 p_3 + c p_1 p_3^3,\ p_{>3} = q_{>3} = 0;$
$U_8^7$: $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_1 p_3^2 = 0,\ q_1 = q_2 = 0, q_3 = -p_1 p_3^3,\ p_{>3} = q_{>3} = 0;,$
$U_8^8$: $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_1 p_3^2 = 0,\ q_{\geq 1} = p_{\geq 4} = 0.$

### 4.1. Distinguishing symplectic classes of $U_8$ by the Lagrangian tangency orders.

A curve $N \in (U_8)$ may be described as a union of three parametrical branches $B_1, B_2$ and $B_3$. Branches $B_1, B_2$ are smooth and their union is an invariant component diffeomorphic to $A_1$ singularity and the branch $B_3$ is diffeomorphic to $A_2$ singularity. Their parametrizations are given in Table 3. To characterize the symplectic classes we use the following invariants:

- $Lt = Lt(B_1, B_2, B_3) = \max\limits_{\mathcal{L}}(\min\{t(B_1, \mathcal{L}), t(B_2, \mathcal{L}), t(B_3, \mathcal{L})\})$,
- $L_{1,2} = Lt(B_1, B_2) = \max\limits_{\mathcal{L}}(\min\{t(B_1, \mathcal{L}), t(B_2, \mathcal{L})\})$,
- $L_3 = Lt(B_3) = \max\limits_{\mathcal{L}} t(B_3, \mathcal{L})$,

Here $L$ is a smooth Lagrangian submanifold of the symplectic space.

**Theorem 4.2.** *Any stratified submanifold $N \in (U_8)$ of a symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with the canonical coordinates $(p_1, q_1, \cdots, p_n, q_n)$ is symplectically equivalent to one and only one of the curves presented in the second column of Table 3. The index of isotropy of the curve $N$ and the Lagrangian tangency orders are presented in the third and fourth, fifth and sixth column of Table 3.*

| class | parametrization of branches of $N$ | | $ind$ | $Lt$ | $L_{1,2}$ | $L_3$ |
|---|---|---|---|---|---|---|
| $(U_8)^0$ | $B_1 : (0, 0, t, 0, 0, \cdots),\ B_2 : (0, t, 0, c_1 t, 0, \cdots)$ | $c_1 \neq 0$ | 0 | 1 | 1 | 3 |
| $2n \geq 4$ | $B_3 : (t^3, t^2, -t^4, c_1 t^2 - c_2 t^3, 0, \cdots)$ | $c_1 = 0$ | 0 | 3 | $\infty$ | 3 |
| $(U_8)^1$ | $B_1 : (0, c_1 t, t, 0, 0, \cdots),\ B_2 : (0, \frac{c_2}{2} t^2, 0, \pm t, 0, \cdots)$ | $c_2 \neq 2c_1$ | 0 | 1 | 1 | 5 |
| $2n \geq 4$ | $B_3 : (t^3, (\frac{c_2}{2} - c_1) t^4, -t^4, \pm t^2, 0, \cdots)$ | $c_2 = 2c_1$ | 0 | 1 | 1 | $\infty$ |
| $(U_8)^2$ | $B_1 : (0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, t, 0, \cdots)$ | $c_1 \neq 2,$ | 0 | 3 | $\infty$ | 5 |
| $2n \geq 4$ | $B_3 : (t^3, -t^4, c_1 t^5 + \frac{c_2}{2} t^6, t^2, 0, \cdots)$ | $c_1 \neq -\frac{1}{3}$ | | | | |
| $(U_8)^{3,0}_5$ | $B_1 : (0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, t, 0, \cdots)$ | | 0 | 3 | $\infty$ | 5 |
| $2n \geq 4$ | $B_3 : (t^3, -t^4, -\frac{1}{3} t^5 + \frac{c_1}{2} t^6 + c_2 t^7, t^2, 0, \cdots)$ | | | | | |
| $(U_8)^{3,0}_\infty$ | $B_1 : (0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, t, 0, \cdots)$ | | 0 | 3 | $\infty$ | $\infty$ |
| $2n \geq 4$ | $B_3 : (t^3, -t^4, 2t^5 + \frac{c_1}{2} t^6 + \frac{c_2}{2} t^8, t^2, 0, \cdots)$ | | | | | |
| $(U_8)^{3,1}$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, 0, t, 0, \cdots)$ | | 1 | 5 | $\infty$ | 5 |
| $2n \geq 6$ | $B_3 : (t^3, 0, -t^4, 0, t^2, -t^5 - \frac{c}{2} t^6, 0, \cdots)$ | | | | | |
| $(U_8)^4$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, 0, t, 0, \cdots)$ | | 1 | 6 | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_3 : (t^3, \pm t^5, -t^4, 0, t^2, -ct^7, 0, \cdots)$ | | | | | |
| $(U_8)^5$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, 0, t, 0, \cdots)$ | | 2 | 7 | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_3 : (t^3, 0, -t^4, 0, t^2, -t^7 - \frac{c}{2} t^8, 0, \cdots)$ | | | | | |
| $(U_8)^6$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, 0, t, 0, \cdots)$ | | 2 | 8 | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_3 : (t^3, 0, -t^4, 0, t^2, \mp \frac{1}{2} t^8 + ct^9, 0, \cdots)$ | | | | | |
| $(U_8)^7$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, 0, t, 0, \cdots)$ | | 3 | 9 | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_3 : (t^3, 0, -t^4, 0, t^2, -t^9, 0, \cdots)$ | | | | | |
| $(U_8)^8$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),\ B_2 : (0, 0, 0, 0, t, 0, \cdots)$ | | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_3 : (t^3, 0, -t^4, 0, t^2, 0, 0, \cdots)$ | | | | | |

TABLE 3. The symplectic invariants for symplectic classes of $U_8$ singularity.

**Remark.** The comparison of invariants presented in Table 3 shows that the Lagrangian tangency order distinguishes more symplectic classes than the index of isotropy. Symplectic classes $(U_8)^2$ and $(U_8)^{3,0}_5$ can be distinguished by the symplectic multiplicity.

The invariants can be calculated by knowing algebraic restrictions for the symplectic classes. We use Proposition 6.5 to calculate the index of isotropy. The invariants $L_{1,2}$ and $L_3$ we can calculate knowing the respective Lagrangian tangency orders for $A_1$ and $A_2$ singularities. $Lt$ is computed by applying directly the definition of the Lagrangian tangency order and finding a Lagrangian submanifold the nearest to the curve $N$.

4.2. **Geometric conditions for the classes $(U_8)^i$.**
We can characterize the symplectic classes $(U_8)^i$ by geometric conditions independent of any local coordinate system.

Let $N \in (U_8)$. Denote by $W$ the tangent space at 0 to some non-singular 3-manifold containing $N$. We can define the following subspaces of this space:

$\ell_1$ – the tangent line at 0 to the nonsingular branch $B_1$,

$\ell_2$ – the tangent line at 0 to the nonsingular branch $B_2$ (this line is also tangent at 0 to the singular branch $B_3$),

$V$ – the 2-space tangent at 0 to the singular branch $B_3$.

For $N = U_8$ =(4.1) it is easy to calculate that $W = \text{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, and $\ell_1 = \text{span}(\partial/\partial x_2)$, $\ell_2 = \text{span}(\partial/\partial x_3)$, $V = \text{span}(\partial/\partial x_1, \partial/\partial x_3)$.

The classes $(U_8)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where $\omega$ is the symplectic form.

**Theorem 4.3.** *If a stratified submanifold $N \in (U_8)$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ belongs to the class $(U_8)^i$ then the couple $(N, \omega)$ satisfies the corresponding conditions in the last column of Table 4.*

| class | normal form | geometric conditions |
|---|---|---|
| $(U_8)^0$ | $[U_8]^0_1 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_8},\ c_1 \neq 0$ | $\omega\|_V \neq 0$ and $\omega\|_{\ell_1+\ell_2} \neq 0$ |
| | $[U_8]^0_\infty : [\theta_1 + c_2\theta_3]_{U_8}$ | $\omega\|_V \neq 0$ and $\omega\|_{\ell_1+\ell_2} = 0$ |
| $(U_8)^1$ | $[U_8]^1_5 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{U_8},\ c_2 \neq 2c_1$ | $\omega\|_V = 0$, $\omega\|_{\ell_1+\ell_2} \neq 0$ and $L_3 = 5$ |
| | $[U_8]^1_\infty : [\pm\theta_2 + c_1\theta_3 + 2c_1\theta_4]_{U_8}$ | $\omega\|_V = 0$, $\omega\|_{\ell_1+\ell_2} \neq 0$ and $L_3 = \infty$ |
| $(U_8)^2$ | $[U_8]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{U_8}$, $c_1 \neq 2$, $c_1 \neq -\frac{1}{3}$ | $\ker\omega = \ell_2$ and $L_3 = 5$ |
| $(U_8)^{3,0}_5$ | $[U_8]^{3,0}_5 : [\theta_3 - \frac{1}{3}\theta_4 + c_1\theta_5 + c_2\theta_6]_{U_8}$ | $\ker\omega = \ell_2$ and $L_3 = 5$ |
| $(U_8)^{3,0}_\infty$ | $[U_8]^{3,0}_\infty : [\theta_3 + 2\theta_4 + c_1\theta_5 + c_2\theta_7]_{U_8}$ | $\ker\omega = \ell_2$ and $L_3 = \infty$ |
| | | $\omega\|_W = 0$ and $L_{1,2} = \infty$ |
| $(U_8)^{3,1}$ | $[U_8]^{3,1} : [\theta_4 + c\theta_5]_{U_8}$ | $Lt = L_3 = 5$ |
| $(U_8)^4$ | $[U_8]^4 : [\pm\theta_5 + c\theta_6]_{U_8}$ | $Lt = 6,\ L_3 = \infty$ |
| $(U_8)^5$ | $[U_8]^5 : [\theta_6 + c\theta_7]_{U_8}$ | $Lt = 7,\ L_3 = \infty$ |
| $(U_8)^6$ | $[U_8]^6 : [\pm\theta_7 + c\theta_8]_{U_8}$ | $Lt = 8,\ L_3 = \infty$ |
| $(U_8)^7$ | $[U_8]^7 : [\theta_8]_{U_8}$ | $Lt = 9,\ L_3 = \infty$ |
| $(U_8)^8$ | $[U_8]^8 : [0]_{U_8}$ | $N$ is contained in a smooth Lagrangian submanifold |

TABLE 4. Geometric interpretation of singularity classes of $U_8$. ($W$ is the tangent space to a non-singular 3-dimensional manifold in $(\mathbb{R}^{2n\geq 4}, \omega)$ containing $N \in (U_8)$. The forms $\theta_1, \ldots, \theta_8$ are described in Theorem 6.19 on the page 18.)

**Remark.** The idea of the proof of Theorem 4.3 is the same as for the proof of Theorem 3.3.

## 5. SYMPLECTIC $U_9$-SINGULARITIES

Denote by $(U_9)$ the class of varieties in a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ which are diffeomorphic to

$$U_9 = \{x \in \mathbb{R}^{2n\geq 4} : x_1^2 + x_2x_3 = x_1x_2 + x_3^4 = x_{\geq 4} = 0\}. \tag{5.1}$$

This is the simple 1-dimensional isolated complete intersection singularity $U_9$ ([G], [AVG]). Here $N$ is quasi-homogeneous with weights $w(x_1)=5$, $w(x_2)=7$, $w(x_3)=3$.

The complete classification of symplectic singularities of $(U_9)$ was obtained using the method of algebraic restrictions.

**Theorem 5.1.** *Any submanifold of the symplectic space* $(\mathbb{R}^{2n}, \sum_{i=1}^n dp_i \wedge dq_i)$ *where* $n \geq 3$ *(respectively* $n = 2$*) which is diffeomorphic to* $U_9$ *is symplectically equivalent to one and only one of the normal forms* $U_9^i, i = 0, 1, \cdots, 9$ *listed below. The parameters* $c, c_1, c_2, c_3$ *of the normal forms are moduli:*

$U_9^0$ : $p_1^2 + p_2 q_1 = 0, \quad \pm p_1 p_2 + q_1^4 = 0, \quad q_2 = c_1 q_1 \mp c_2 p_1, \quad p_{\geq 3} = q_{\geq 3} = 0$;

$U_9^1$ : $p_2^2 \pm p_1 q_1 = 0, \quad p_1 p_2 + q_1^4 = 0, \quad q_2 = c_1 p_1 + \frac{c_2}{2} q_1^2 \pm \frac{c_3}{3} q_1^3, \quad p_{\geq 3} = q_{\geq 3} = 0$;

$U_9^2$ : $p_1^2 \pm q_1 q_2 = 0, \quad \pm p_1 q_1 + q_2^4 = 0, \quad p_2 = c_1 p_1 q_2 + \frac{c_2}{2} p_1^2, \quad p_{\geq 3} = q_{\geq 3} = 0,\ c_1 \neq 0$;

$U_9^{3,0}$ : $p_1^2 \pm q_1 q_2 = 0, \quad \pm p_1 q_1 + q_2^4 = 0, \quad p_2 = \frac{c_1}{2} p_1^2 + c_2 p_1 q_2^2, \quad p_{\geq 3} = q_{\geq 3} = 0,\ c_1 \neq 0$;

$U_9^{4,0}$ : $p_1^2 \pm q_1 q_2 = 0, \quad \pm p_1 q_1 + q_2^4 = 0, \quad p_2 = c_1 p_1 q_2^2 + \frac{c_2}{2} p_1^2 q_2, \quad p_{\geq 3} = q_{\geq 3} = 0$;

$U_9^{3,1}$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_1 = q_2 = 0, q_3 = -p_1 p_3 - \frac{c}{2} p_1^2,\ p_{>3} = q_{>3} = 0$;

$U_9^{4,1}$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_1 = q_2 = 0, q_3 = -\frac{1}{2} p_1^2 - c_1 p_1 p_3^2 - c_2 p_1 p_3^3$,
$p_{>3} = q_{>3} = 0$;

$U_9^5$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_1 = q_2 = 0, q_3 = \mp p_1 p_3^2 - \frac{c}{2} p_1^2 p_3,\ p_{>3} = q_{>3} = 0$;

$U_9^6$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_1 = q_2 = 0, q_3 = \mp \frac{1}{2} p_1^2 p_3 - c p_1 p_3^3,\ p_{>3} = q_{>3} = 0$;

$U_9^7$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_1 = q_2 = 0, q_3 = -p_1 p_3^3 - \frac{c}{2} p_1^2 p_3^2,\ p_{>3} = q_{>3} = 0$;

$U_9^8$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_1 = q_2 = 0, q_3 = -\frac{1}{2} p_1^2 p_3^2,\ p_{>3} = q_{>3} = 0$;

$U_9^9$ : $p_1^2 + p_2 p_3 = 0,\ p_1 p_2 + p_3^4 = 0,\ q_{\geq 1} = p_{\geq 4} = 0$.

5.1. **Distinguishing symplectic classes of $U_9$ by Lagrangian tangency orders.** The Lagrangian tangency orders were used to distinguish the symplectic classes of $(U_9)$. A curve $N \in (U_9)$ may be described as a union of two parametrical branches: $B_1$ and $B_2$. The curve $B_1$ is nonsingular and the curve $B_2$ is singular. Their parametrization in the coordinate system $(p_1, q_1, p_2, q_2, \cdots, p_n, q_n)$ is presented in the second column of Table 5. To characterize the symplectic classes of this singularity we use the following two invariants:

- $Lt = Lt(B_1, B_2) = \max_L(\min\{t(B_1, L), t(B_2, L)\})$,
- $L_2 = Lt(B_2) = \max_L t(B_2, L)$.

Here $L$ is a smooth Lagrangian submanifold of the symplectic space.

We can also compare the Lagrangian tangency orders with the respective indices of isotropy.

**Theorem 5.2.** *A stratified submanifold* $N \in (U_9)$ *of the symplectic space* $(\mathbb{R}^{2n}, \omega_0)$ *with the canonical coordinates* $(p_1, q_1, \cdots, p_n, q_n)$ *is symplectically equivalent to one and only one of the curves presented in the second column of Table 5. The parameters* $c, c_1, c_2, c_3$ *are moduli. The Lagrangian tangency orders are presented in the third and fourth column of Table 5.*

| class | parametrization of branches | | $ind$ | $ind_2$ | $Lt$ | $L_2$ |
|---|---|---|---|---|---|---|
| $(U_9)^0$ | $B_1 : (0, 0, t, 0, 0, \cdots),$ | $c_1 \neq 0$ | 0 | 0 | 3 | 5 |
| $2n \geq 4$ | $B_2 : (\pm t^5, t^3, -t^7, c_1 t^3 - c_2 t^5, 0, \cdots)$ | $c_1 = 0$ | 0 | 0 | 5 | 5 |
| $(U_9)^1$ | $B_1 : (t, 0, 0, c_1 t, 0, \cdots),$ | | 0 | 0 | 3 | 7 |
| $2n \geq 4$ | $B_2 : (-t^7, \pm t^3, t^5, -c_1 t^7 + \frac{c_2}{2} t^6 \pm \frac{c_3}{3} t^9, 0, \cdots)$ | | | | | |
| $(U_9)^2$ | $B_1 : (0, \pm t, 0, 0, 0, \cdots),$ | $c_1 \neq 0$ | 0 | 0 | 5 | 7 |
| $2n \geq 4$ | $B_2 : (t^5, \mp t^7, c_1 t^8 + \frac{c_2}{2} t^{10}, t^3, 0, \cdots)$ | | | | | |
| $(U_9)^{3,0}$ | $B_1 : (0, \pm t, 0, 0, 0, \cdots),$ | $c_1 \neq 0$ | 0 | 0 | 5 | 7 |
| $2n \geq 4$ | $B_2 : (t^5, \mp t^7, \frac{c_1}{2} t^{10} + c_2 t^{11}, t^3, 0, \cdots)$ | | | | | |
| $(U_9)^{4,0}$ | $B_1 : (0, \pm t, 0, 0, 0, \cdots),$ | | 0 | 0 | 5 | 7 |
| $2n \geq 4$ | $B_2 : (t^5, \mp t^7, c_1 t^{11} + \frac{c_2}{2} t^{13}, t^3, 0, \cdots)$ | | | | | |
| $(U_9)^{3,1}$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | 1 | 1 | 8 | 8 |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, -t^8 - \frac{c}{2} t^{10}, 0, \cdots)$ | | | | | |
| $(U_9)^{4,1}$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | 1 | 1 | 10 | 10 |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, -\frac{1}{2} t^{10} - c_1 t^{11} - c_2 t^{14}, 0, \cdots)$ | | | | | |
| $(U_9)^5$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | 2 | 2 | 11 | 11 |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, \mp t^{11} - \frac{c}{2} t^{13}, 0, \cdots)$ | | | | | |
| $(U_9)^6$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | 2 | 2 | 13 | 13 |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, \mp \frac{1}{2} t^{13} - c t^{14}, 0, \cdots)$ | | | | | |
| $(U_9)^7$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | 3 | $\infty$ | 14 | $\infty$ |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, -t^{14} - \frac{c}{2} t^{16}, 0, \cdots)$ | | | | | |
| $(U_9)^8$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | 3 | $\infty$ | 16 | $\infty$ |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, -\frac{1}{2} t^{16}, 0, \cdots)$ | | | | | |
| $(U_9)^9$ | $B_1 : (0, 0, t, 0, 0, 0, \cdots),$ | | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| $2n \geq 6$ | $B_2 : (t^5, 0, -t^7, 0, t^3, 0, 0, \cdots)$ | | | | | |

TABLE 5. The Lagrangian tangency orders for symplectic classes of the $U_9$ singularity.

5.2. **Geometric conditions for the classes $(U_9)^i$.**

Let $N \in (U_9)$. Denote by $W$ the tangent space at 0 to some (and then any) non-singular 3-manifold containing $N$. We can define the following subspaces of this space:

$\ell_1$ – the tangent line at 0 to the nonsingular branch $B_1$,

$\ell_2$ – the tangent line at 0 to the singular branch $B_2$,

$V$ – the 2-space tangent at 0 to the singular branch $B_2$.

For $N = U_9 =$(5.1) it is easy to calculate that $W = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, and $\ell_1 = \mathrm{span}(\partial/\partial x_2)$, $\ell_2 = \mathrm{span}(\partial/\partial x_3)$, $V = \mathrm{span}(\partial/\partial x_1, \partial/\partial x_3)$.

The classes $(U_9)^i$ satisfy special conditions in terms of the restriction $\omega|_W$, where $\omega$ is the symplectic form.

**Theorem 5.3.** *For any stratified submanifold $N \in (U_9)$ of the symplectic space $(\mathbb{R}^{2n}, \omega)$ belonging to the class $(U_9)^i$ the couple $(N, \omega)$ satisfies the corresponding conditions in the last column of Table 6.*

| class | normal form | geometric conditions |
|---|---|---|
| $(U_9)^0$ | $[U_9]^0_0 : [\pm\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_9},\ c_1 \neq 0$ | $\omega\vert_V \neq 0$ and $\omega\vert_{\ell_1+\ell_2} \neq 0$ |
| | $[U_9]^0_1 : [\pm\theta_1 + c_2\theta_3]_{U_9}$ | $\omega\vert_V \neq 0$ and $\omega\vert_{\ell_1+\ell_2} = 0$ |
| $(U_9)^1$ | $[U_9]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4 + c_3\theta_6]_{U_9}$ | $\omega\vert_V = 0$, $\omega\vert_{\ell_1+\ell_2} = 0$ and $\ker\omega \neq \ell_2$ |
| $(U_9)^2$ | $[U_9]^2 : [\pm\theta_3 + c_1\theta_4 + c_2\theta_5]_{U_9},\ c_1 \neq 0$ | $\omega\vert_V = 0$ and $\ker\omega = \ell_2$ |
| $(U_9)^{3,0}$ | $[U_9]^{3,0} : [\pm\theta_3 + c_1\theta_5 + c_2\theta_6]_{U_9},\ c_1 \neq 0$ | |
| $(U_9)^{4,0}$ | $[U_9]^{4,0} : [\pm\theta_3 + c_1\theta_6 + c_2\theta_7]_{U_9}$ | |
| | | $\omega\vert_W = 0$ |
| $(U_9)^{3,1}$ | $[U_9]^{3,1} : [\theta_4 + c\theta_5]_{U_9}$ | $Lt = L_2 = 8$ |
| $(U_9)^{4,1}$ | $[U_9]^{4,1} : [\theta_5 + c_1\theta_6 + c_2\theta_8]_{U_9}$ | $Lt = L_2 = 10$ |
| $(U_9)^5$ | $[U_9]^5 : [\pm\theta_6 + c\theta_7]_{U_9}$ | $Lt = L_2 = 11$ |
| $(U_9)^6$ | $[U_9]^6 : [\pm\theta_7 + c\theta_8]_{U_9}$ | $Lt = L_2 = 13$ |
| $(U_9)^7$ | $[U_9]^7 : [\theta_8 + c\theta_9]_{U_9}$ | $Lt = 14,\ L_2 = \infty$ |
| $(U_9)^8$ | $[U_9]^8 : [\theta_9]_{U_9}$ | $Lt = 16,\ L_2 = \infty$ |
| $(U_9)^9$ | $[U_9]^9 : [0]_{U_9}$ | $N$ is contained in a smooth Lagrangian submanifold |

TABLE 6. Geometric characterization of symplectic classes of the $U_9$ singularity. (The forms $\theta_1, \dots, \theta_9$ are described in Theorem 6.33 on the page 21.)

**Remark.** The idea of the proof of Theorem 5.3 is the same as for the proof of Theorem 3.3.

## 6. Proofs

6.1. **The method of algebraic restrictions.** In this section we present only basic notions and facts on the method of algebraic restrictions, which is a very powerful tool for the symplectic classification. The details of the method can be found in [DJZ2].

Given a germ of a non-singular manifold $M$ denote by $\Lambda^p(M)$ the space of all germs at 0 of differential $p$-forms on $M$. Given a subset $N \subset M$ introduce the following subspaces of $\Lambda^p(M)$:

$$\Lambda^p_N(M) = \{\omega \in \Lambda^p(M) : \quad \omega(x) = 0 \text{ for any } x \in N\};$$
$$\mathcal{A}^p_0(N, M) = \{\alpha + d\beta : \quad \alpha \in \Lambda^p_N(M),\ \beta \in \Lambda^{p-1}_N(M)\}.$$

**Definition 6.1.** Let $N$ be the germ of a subset of $M$ and let $\omega \in \Lambda^p(M)$. The *algebraic restriction* of $\omega$ to $N$ is the equivalence class of $\omega$ in $\Lambda^p(M)$, where the equivalence is as follows: $\omega$ is equivalent to $\widetilde{\omega}$ if $\omega - \widetilde{\omega} \in \mathcal{A}^p_0(N, M)$.

**Notation.** The algebraic restriction of the germ of a $p$-form $\omega$ on $M$ to the germ of a subset $N \subset M$ will be denoted by $[\omega]_N$. By writing $[\omega]_N = 0$ (or saying that $\omega$ has zero algebraic restriction to $N$) we mean that $[\omega]_N = [0]_N$, i.e. $\omega \in A^p_0(N, M)$.

**Definition 6.2.** Two algebraic restrictions $[\omega]_N$ and $[\widetilde{\omega}]_{\widetilde{N}}$ are called *diffeomorphic* if there exists the germ of a diffeomorphism $\Phi : \widetilde{M} \to M$ such that $\Phi(\widetilde{N}) = N$ and $\Phi^*([\omega]_N) = [\widetilde{\omega}]_{\widetilde{N}}$.

The method of algebraic restrictions applied to singular quasi-homogeneous subsets is based on the following theorem.

**Theorem 6.1** (Theorem A in [DJZ2]). *Let $N$ be the germ of a quasi-homogeneous subset of $\mathbb{R}^{2n}$. Let $\omega_0, \omega_1$ be germs of symplectic forms on $\mathbb{R}^{2n}$ with the same algebraic restriction to $N$. There exists a local diffeomorphism $\Phi$ such that $\Phi(x) = x$ for any $x \in N$ and $\Phi^*\omega_1 = \omega_0$.*

*Two germs of quasi-homogeneous subsets $N_1, N_2$ of a fixed symplectic space $(\mathbb{R}^{2n}, \omega)$ are symplectically equivalent if and only if the algebraic restrictions of the symplectic form $\omega$ to $N_1$ and $N_2$ are diffeomorphic.*

Theorem 6.1 reduces the problem of symplectic classification of germs of singular quasi-homogeneous subsets to the problem of diffeomorphic classification of algebraic restrictions of the germ of the symplectic form to the germs of singular quasi-homogeneous subsets.

The geometric meaning of the zero algebraic restriction is explained by the following theorem.

**Theorem 6.2** (Theorem **B** in [DJZ2]). *The germ of a quasi-homogeneous set $N$ of a symplectic space $(\mathbb{R}^{2n}, \omega)$ is contained in a non-singular Lagrangian submanifold if and only if the symplectic form $\omega$ has zero algebraic restriction to $N$.*

In the remainder of this paper we use the following notations:

- $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$: the vector space consisting of the algebraic restrictions of germs of all 2-forms on $\mathbb{R}^{2n}$ to the germ of a subset $N \subset \mathbb{R}^{2n}$;
- $\left[Z^2(\mathbb{R}^{2n})\right]_N$: the subspace of $\left[\Lambda^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all closed 2-forms on $\mathbb{R}^{2n}$ to $N$;
- $\left[\mathrm{Symp}(\mathbb{R}^{2n})\right]_N$: the open set in $\left[Z^2(\mathbb{R}^{2n})\right]_N$ consisting of the algebraic restrictions of germs of all symplectic 2-forms on $\mathbb{R}^{2n}$ to $N$.

To obtain a classification of the algebraic restrictions we use the following proposition.

**Proposition 6.3.** *Let $a_1, \cdots, a_p$ be a quasi-homogeneous basis of quasi-degrees $\delta_1 \leq \cdots \leq \delta_s < \delta_{s+1} \leq \cdots \leq \delta_p$ of the space of algebraic restrictions of closed 2-forms to quasi-homogeneous subset $N$. Let $a = \sum_{j=s}^{p} c_j a_j$, where $c_j \in \mathbb{R}$ for $j = s, \cdots, p$ and $c_s \neq 0$.*

*If there exists a tangent quasi-homogeneous vector field $X$ over $N$ such that $\mathcal{L}_X a_s = r a_k$ for $k > s$ and $r \neq 0$ then $a$ is diffeomorphic to $\sum_{j=s}^{k-1} c_j a_j + \sum_{j=k+1}^{p} b_j a_j$, for some $b_j \in \mathbb{R}$, $j = k+1, \cdots, p$.*

Proposition 6.3 is a modification of Theorem 6.13 formulated and proved in [**?**]. It was formulated for algebraic restrictions to a parameterized curve but we can generalize this theorem for any quasi-homogeneous subset $N$. The proofs of the cited theorem and Proposition 6.3 are based on the Moser homotopy method.

For calculating discrete invariants we use the following propositions.

**Proposition 6.4** ([DJZ2]). *The symplectic multiplicity of the germ of a quasi-homogeneous subset $N$ in a symplectic space is equal to the codimension of the orbit of the algebraic restriction $[\omega]_N$ with respect to the group of local diffeomorphisms preserving $N$ in the space of algebraic restrictions of closed 2-forms to $N$.*

**Proposition 6.5** ([DJZ2]). *The index of isotropy of the germ of a quasi-homogeneous subset $N$ in a symplectic space $(\mathbb{R}^{2n}, \omega)$ is equal to the maximal order of vanishing of closed 2-forms representing the algebraic restriction $[\omega]_N$.*

**Proposition 6.6** ([?]). *Let $f$ be the germ of a quasi-homogeneous curve such that the algebraic restriction of a symplectic form to it can be represented by a closed 2-form vanishing at 0. Then the Lagrangian tangency order of the germ of a quasi-homogeneous curve $f$ is the maximum of the order of vanishing on $f$ over all 1-forms $\alpha$ such that $[\omega]_f = [d\alpha]_f$*

6.2. **Proofs for $U_7$ singularity.**

6.2.1. *Algebraic restrictions to $U_7$ and their classification.* One has the following relations for $(U_7)$-singularities:

$$[x_1^2 + x_2x_3]_{U_7} = 0. \tag{6.1}$$

$$[x_1x_2 + x_3^3]_{U_7} = 0, \tag{6.2}$$

$$[d(x_1^2 + x_2x_3)]_{U_7} = [2x_1dx_1 + x_2dx_3 + x_3dx_2]_{U_7} = 0 \tag{6.3}$$

$$[d(x_1x_2 + x_3^3)]_{U_7} = [x_1dx_2 + x_2dx_1 + 3x_3^2dx_3]_{U_7} = 0 \tag{6.4}$$

Multiplying these relations by suitable 1-forms and 2-forms we obtain the relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = U_7$.

**Proposition 6.7.** *The space $[\Lambda^2(\mathbb{R}^{2n})]_{U_7}$ is an 8-dimensional vector space spanned by the algebraic restrictions to $U_7$ of the 2-forms*
$\theta_1 = dx_1 \wedge dx_3$, $\theta_2 = dx_2 \wedge dx_3$, $\theta_3 = dx_1 \wedge dx_2$,
$\theta_4 = x_3dx_1 \wedge dx_3$, $\theta_5 = x_1dx_1 \wedge dx_3$, $\sigma = x_1dx_2 \wedge dx_3$,
$\theta_6 = x_3^2dx_1 \wedge dx_3$, $\theta_7 = x_1x_3dx_1 \wedge dx_3$.

Proposition 6.7 and results of Section 6.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{U_7}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{U_7}$.

**Theorem 6.8.** *The space $[Z^2(\mathbb{R}^{2n})]_{U_7}$ is a 7-dimensional vector space spanned by the algebraic restrictions to $U_7$ of the quasi-homogeneous 2-forms $\theta_i$ of degree $\delta_i$*
$\theta_1 = dx_1 \wedge dx_3$, $\delta_1 = 7$,
$\theta_2 = dx_2 \wedge dx_3$, $\delta_2 = 8$,
$\theta_3 = dx_1 \wedge dx_2$, $\delta_3 = 9$,
$\theta_4 = x_3dx_1 \wedge dx_3$, $\delta_4 = 10$,
$\theta_5 = x_1dx_1 \wedge dx_3$, $\delta_5 = 11$,
$\theta_6 = x_3^2dx_1 \wedge dx_3$, $\delta_6 = 13$,
$\theta_7 = x_1x_3dx_1 \wedge dx_3$, $\delta_7 = 14$.

*If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{U_7} = [Z^2(\mathbb{R}^{2n})]_{U_7}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{U_7}$ is an open part of the 7-space $[Z^2(\mathbb{R}^4)]_{U_7}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_7\theta_7]_{U_7}$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$.*

**Theorem 6.9.**
*(i) Any algebraic restriction in* $[Z^2(\mathbb{R}^{2n})]_{U_7}$ *can be brought by a symmetry of* $U_7$ *to one of the normal forms* $[U_7]^i$ *given in the second column of Table 7.*
*(ii) The codimension in* $[Z^2(\mathbb{R}^{2n})]_{U_7}$ *of the singularity class corresponding to the normal form* $[U_7]^i$ *is equal to* $i$*, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 7.*
*(iii) The singularity classes corresponding to the normal forms are disjoint.*
*(iv) The parameters* $c, c_1, c_2$ *of the normal forms* $[U_7]^i$ *are moduli.*

| symplectic class | normal forms for algebraic restrictions | cod | $\mu^{\mathrm{sym}}$ | ind |
|---|---|---|---|---|
| $(U_7)^0$ $(2n \geq 4)$ | $[U_7]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_7}$, | 0 | 2 | 0 |
| $(U_7)^1$ $(2n \geq 4)$ | $[U_7]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{U_7}$ | 1 | 3 | 0 |
| $(U_7)^2$ $(2n \geq 4)$ | $[U_7]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{U_7}$, | 2 | 4 | 0 |
| $(U_7)^3$ $(2n \geq 6)$ | $[U_7]^3 : [\pm\theta_4 + c\theta_5]_{U_7}$ | 3 | 4 | 1 |
| $(U_7)^4$ $(2n \geq 6)$ | $[U_7]^4 : [\theta_5 + c\theta_6]_{U_7}$ | 4 | 5 | 1 |
| $(U_7)^5$ $(2n \geq 6)$ | $[U_7]^5 : [\theta_6 + c\theta_7]_{U_7}$ | 5 | 6 | 2 |
| $(U_7)^6$ $(2n \geq 6)$ | $[U_7]^6 : [\pm\theta_7]_{U_7}$ | 6 | 6 | 2 |
| $(U_7)^7$ $(2n \geq 6)$ | $[U_7]^7 : [0]_{U_7}$ | 7 | 7 | $\infty$ |

TABLE 7. Classification of symplectic $U_7$ singularities.
*cod* – codimension of the classes; $\mu^{sym}$– symplectic multiplicity;
*ind* – the index of isotropy.

In the first column of Table 7 we denote by $(U_7)^i$ a subclass of $(U_7)$ consisting of $N \in (U_7)$ such that the algebraic restriction $[\omega]_N$ is diffeomorphic to some algebraic restriction of the normal form $[U_7]^i$, where $i$ is the codimension of the class.

The proof of Theorem 6.9 is presented in Section 6.2.3.

6.2.2. *Symplectic normal forms.* Let us transfer the normal forms $[U_7]^i$ to symplectic normal forms. Fix a family $\omega^i$ of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[U_7]^i$ of algebraic restrictions. We can fix, for example,

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^1 = \pm\theta_2 + c_1\theta_3 + c_2\theta_4 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^2 = \theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^3 = \pm\theta_4 + c\theta_5 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^4 = \theta_5 + c\theta_6 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^5 = \theta_6 + c\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^6 = \pm\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^7 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

Let $\omega_0 = \sum_{i=1}^m dp_i \wedge dq_i$, where $(p_1, q_1, \cdots, p_n, q_n)$ is the coordinate system on $\mathbb{R}^{2n}, n \geq 3$ (resp. $n = 2$). Fix, for $i = 0, 1, \cdots, 7$ (resp. for $i = 0, 1, 2$) a family $\Phi^i$ of local diffeomorphisms which bring the family of symplectic forms $\omega^i$ to the symplectic form $\omega_0$: $(\Phi^i)^*\omega^i = \omega_0$. Consider the families $U_7^i = (\Phi^i)^{-1}(U_7)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to

$U_7$ is symplectically equivalent to one and only one of the normal forms $U_7^i, i = 0, 1, \cdots, 7$ (resp. $i = 0, 1, 2$) presented in Theorem 3.1. By Theorem 6.9 we obtain that parameters $c, c_1, c_2$ of the normal forms are moduli.

6.2.3. *Proof of Theorem 6.9.* In our proof we use vector fields tangent to $N \in U_7$. Any vector fields tangent to $N \in U_7$ can be described as $V = g_1 E + g_2 \mathcal{H}$ where $E$ is the Euler vector field and $\mathcal{H}$ is a Hamiltonian vector field and $g_1, g_2$ are functions. It was shown in [DT1] (Prop. 6.13) that the action of a Hamiltonian vector field on the algebraic restriction of a closed 2-form to any 1-dimensional complete intersection is trivial.

The germ of a vector field tangent to $U_7$ of non trivial action on algebraic restrictions of closed 2-forms to $U_7$ may be described as a linear combination of germs of vector fields: $X_0 = E$, $X_1 = x_3 E$, $X_2 = x_1 E$, $X_3 = x_2 E$, $X_4 = x_3 E^2$, $X_5 = x_1 x_3 E$, where $E$ is the Euler vector field

$$E = 4x_1 \partial/\partial x_1 + 5x_2 \partial/\partial x_2 + 3x_3 \partial/\partial x_3. \tag{6.5}$$

**Proposition 6.10.** *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (U_7)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to $N$ is presented in Table 8.*

| $\mathcal{L}_{X_i}[\theta_j]$ | $[\theta_1]$ | $[\theta_2]$ | $[\theta_3]$ | $[\theta_4]$ | $[\theta_5]$ | $[\theta_6]$ | $[\theta_7]$ |
|---|---|---|---|---|---|---|---|
| $X_0 = E$ | $7[\theta_1]$ | $8[\theta_2]$ | $9[\theta_3]$ | $10[\theta_4]$ | $11[\theta_5]$ | $13[\theta_6]$ | $14[\theta_7]$ |
| $X_1 = x_3 E$ | $10[\theta_4]$ | $-22[\theta_5]$ | $[0]$ | $13[\theta_6]$ | $14[\theta_7]$ | $[0]$ | $[0]$ |
| $X_2 = x_1 E$ | $11[\theta_5]$ | $[0]$ | $-39[\theta_6]$ | $14[\theta_7]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_3 = x_2 E$ | $[0]$ | $-78[\theta_6]$ | $-84[\theta_7]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_4 = x_3^2 E$ | $13[\theta_6]$ | $-28[\theta_7]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_5 = x_1 x_3 E$ | $14[\theta_7]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |

Table 8. Infinitesimal actions on algebraic restrictions of closed 2-forms to $U_7$. ($E$ is defined as in (6.5).)

Let $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7]_{U_7}$ be the algebraic restriction of a symplectic form $\omega$.

The first statement of Theorem 6.9 follows from the following lemmas.

**Lemma 6.11.** *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^{7} c_k\theta_k]_{U_7}$ can be reduced by a symmetry of $U_7$ to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{U_7}$.*

**Lemma 6.12.** *If $c_1 = 0$ and $c_2 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_7$ to an algebraic restriction $[\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{U_7}$.*

**Lemma 6.13.** *If $c_1 = c_2 = 0$ and $c_3 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_7$ to an algebraic restriction $[\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{U_7}$.*

**Lemma 6.14.** *If $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_7$ to an algebraic restriction $[\pm\theta_4 + \widetilde{c}_5\theta_5]_{U_7}$.*

**Lemma 6.15.** *If $c_1 = 0, \ldots, c_4 = 0$ and $c_5 \neq 0$, then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_7$ to an algebraic restriction $[\theta_5 + \widetilde{c}_6\theta_6]_{U_7}$.*

**Lemma 6.16.** *If* $c_1 = 0, \dots, c_5 = 0$ *and* $c_6 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_7$ *to an algebraic restriction* $[\theta_6 + \widetilde{c}_7\theta_7]_{U_7}$.

**Lemma 6.17.** *If* $c_1 = 0, \dots, c_6 = 0$ *and* $c_7 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_7$ *to an algebraic restriction* $[\pm\theta_7]_{U_7}$.

The proofs of Lemmas 6.11 – 6.17 are similar and based on Table 8 and Proposition 6.3.

Statement $(ii)$ of Theorem 6.9 follows from the conditions in the proof of part $(i)$ (the codimension) and from Theorem 6.2 and Proposition 6.4 (the symplectic multiplicity) and Proposition 6.5 (the index of isotropy).

To prove statement $(iii)$ of Theorem 6.9 we have to show that singularity classes corresponding to normal forms are disjoint. It is enough to notice that the singularity classes can be distinguished by geometric conditions.

To prove statement $(iv)$ of Theorem 6.9 we have to show that the parameters $c, c_1, c_2$ are moduli in the normal forms. The proofs are very similar in all cases. We consider as an example the normal form with two parameters $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_7}$. From Table 8 we see that the tangent space to the orbit of $[\theta_1+c_1\theta_2+c_2\theta_3]_{U_7}$ at $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_7}$ is spanned by the linearly independent algebraic restrictions $[7\theta_1 + 8c_1\theta_2 + 9c_2\theta_3]_{U_7}$, $[\theta_4]_{U_7}, [\theta_5]_{U_7}, [\theta_6]_{U_7}$ and $[\theta_7]_{U_7}$. Hence, the algebraic restrictions $[\theta_2]_{U_7}$ and $[\theta_3]_{U_7}$ do not belong to it. Therefore, the parameters $c_1$ and $c_2$ are independent moduli in the normal form $[\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_7}$.

## 6.3. **Proofs for $U_8$ singularity.**

6.3.1. *Algebraic restrictions to $U_8$ and their classification.* One has the following relations for $(U_8)$-singularities:

$$[x_1^2 + x_2x_3]_{U_8} = 0. \tag{6.6}$$

$$[x_1x_2 + x_1x_3^2]_{U_8} = 0, \tag{6.7}$$

$$[d(x_1^2 + x_2x_3)]_{U_8} = [2x_1dx_1 + x_2dx_3 + x_3dx_2]_{U_8} = 0 \tag{6.8}$$

$$[d(x_1x_2 + x_1x_3^2)]_{U_8} = [x_1dx_2 + x_2dx_1 + x_3^2dx_1 + 2x_1x_3dx_3]_{U_8} = 0 \tag{6.9}$$

Multiplying these relations by suitable 1-forms and 2-forms we obtain the relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = U_8$.

**Proposition 6.18.** *The space* $[\Lambda^2(\mathbb{R}^{2n})]_{U_8}$ *is a* 9*-dimensional vector space spanned by the algebraic restrictions to* $U_8$ *of the* 2*-forms*
$\theta_1 = dx_1 \wedge dx_3$, $\theta_2 = dx_2 \wedge dx_3$, $\theta_3 = dx_1 \wedge dx_2$,
$\theta_4 = x_3dx_1 \wedge dx_3$, $\theta_5 = x_1dx_1 \wedge dx_3$, $\theta_6 = x_3^2dx_1 \wedge dx_3$, $\sigma = x_1dx_2 \wedge dx_3$,
$\theta_7 = x_1x_3dx_1 \wedge dx_3$, $\theta_8 = x_3^3dx_1 \wedge dx_3$.

Proposition 6.18 and results of Section 6.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{U_8}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{U_8}$.

**Theorem 6.19.** *The space $[Z^2(\mathbb{R}^{2n})]_{U_8}$ is an 8-dimensional vector space spanned by the algebraic restrictions to $U_8$ of the quasi-homogeneous 2-forms $\theta_i$ of degree $\delta_i$*

$\theta_1 = dx_1 \wedge dx_3, \quad \delta_1 = 5,$
$\theta_2 = dx_2 \wedge dx_3, \quad \delta_2 = 6,$
$\theta_3 = dx_1 \wedge dx_2, \quad \delta_3 = 7,$
$\theta_4 = x_3 dx_1 \wedge dx_3, \quad \delta_4 = 7,$
$\theta_5 = x_1 dx_1 \wedge dx_3, \quad \delta_5 = 8,$
$\theta_6 = x_3^2 dx_1 \wedge dx_3, \quad \delta_6 = 9,$
$\theta_7 = x_1 x_3 dx_1 \wedge dx_3, \quad \delta_7 = 10,$
$\theta_8 = x_3^3 dx_1 \wedge dx_3, \quad \delta_8 = 11.$

*If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{U_8} = [Z^2(\mathbb{R}^{2n})]_{U_8}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{U_8}$ is an open part of the 8-space $[Z^2(\mathbb{R}^4)]_{U_8}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_8\theta_8]_{U_8}$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$.*

**Theorem 6.20.**

*(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{U_8}$ can be brought by a symmetry of $U_8$ to one of the normal forms $[U_8]^i$ given in the second column of Table 9.*

*(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{U_8}$ of the singularity class corresponding to the normal form $[U_8]^i$ is equal to $i$, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 9.*

*(iii) The singularity classes corresponding to the normal forms are disjoint.*

*(iv) The parameters $c, c_1, c_2$ of the normal forms $[U_8]^i$ are moduli.*

| symplectic class | normal forms for algebraic restrictions | cod | $\mu^{\mathrm{sym}}$ | ind |
|---|---|---|---|---|
| $(U_8)^0 \quad (2n \geq 4)$ | $[U_8]^0 : [\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_8}$, | 0 | 2 | 0 |
| $(U_8)^1 \quad (2n \geq 4)$ | $[U_8]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4]_{U_8}$ | 1 | 3 | 0 |
| $(U_8)^2 \quad (2n \geq 4)$ | $[U_8]^2 : [\theta_3 + c_1\theta_4 + c_2\theta_5]_{U_8}, \quad c_1 \neq -\frac{1}{3}, c_1 \neq 2$ | 2 | 4 | 0 |
| $(U_8)_5^{3,0} \quad (2n \geq 4)$ | $[U_8]_5^{3,0} : [\theta_3 - \frac{1}{3}\theta_4 + c_1\theta_5 + c_2\theta_6]_{U_8}$ | 3 | 5 | 0 |
| $(U_8)_\infty^{3,0} \quad (2n \geq 4)$ | $[U_8]_\infty^{3,0} : [\theta_3 + 2\theta_4 + c_1\theta_5 + c_2\theta_7]_{U_8}$ | 3 | 5 | 0 |
| $(U_8)^{3,1} \quad (2n \geq 6)$ | $[U_8]^{3,1} : [\theta_4 + c\theta_5]_{U_8}$ | 3 | 4 | 1 |
| $(U_8)^4 \quad (2n \geq 6)$ | $[U_8]^4 : [\pm\theta_5 + c\theta_6]_{U_8}$ | 4 | 5 | 1 |
| $(U_8)^5 \quad (2n \geq 6)$ | $[U_8]^5 : [\theta_6 + c\theta_7]_{U_8}$ | 5 | 6 | 2 |
| $(U_8)^6 \quad (2n \geq 6)$ | $[U_8]^6 : [\pm\theta_7 + c\theta_8]_{U_8}$ | 6 | 7 | 2 |
| $(U_8)^7 \quad (2n \geq 6)$ | $[U_8]^7 : [\theta_8]_{U_8}$ | 7 | 7 | 3 |
| $(U_8)^8 \quad (2n \geq 6)$ | $[U_8]^8 : [0]_{U_8}$ | 8 | 8 | $\infty$ |

TABLE 9. Classification of symplectic $U_8$ singularities.
*cod* – codimension of the classes; $\mu^{sym}$– symplectic multiplicity;
*ind* – the index of isotropy.

The proof of Theorem 6.20 is presented in Section 6.3.3.

6.3.2. *Symplectic normal forms.* Let us transfer the normal forms $[U_8]^i$ to symplectic normal forms. Fix a family $\omega^i$ of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[U_8]^i$ of algebraic restrictions. We can fix, for example,

$\omega^0 = \theta_1 + c_1\theta_2 + c_2\theta_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^1 = \pm\theta_2 + c_1\theta_3 + c_2\theta_4 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^2 = \theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^{3,0}_5 = \theta_3 - \frac{1}{3}\theta_4 + c_1\theta_5 + c_2\theta_6 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^{3,0}_\infty = \theta_3 + 2\theta_4 + c_1\theta_5 + c_2\theta_7 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^{3,1} = \theta_4 + c\theta_5 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^4 = \pm\theta_5 + c\theta_6 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^5 = \theta_6 + c\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^6 = \pm\theta_7 + c\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^7 = \theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^8 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

Fix, for $i = 0, 1, \cdots, 8$ a family $\Phi^i$ of local diffeomorphisms which bring the family of symplectic forms $\omega^i$ to the symplectic form $\omega_0$: $(\Phi^i)^*\omega^i = \omega_0$. Consider the families $U_8^i = (\Phi^i)^{-1}(U_8)$. Any stratified submanifold of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ which is diffeomorphic to $U_8$ is symplectically equivalent to one and only one of the normal forms $U_8^i, i = 0, 1, \cdots, 8$ presented in Theorem 4.1. By Theorem 6.20 we obtain that parameters $c, c_1, c_2$ of the normal forms are moduli.

6.3.3. *Proof of Theorem 6.20.* The germ of a vector field tangent to $U_8$ of non trivial action on algebraic restrictions of closed 2-forms to $U_8$ may be described as a linear combination of germs of vector fields: $X_0 = E$, $X_1 = x_3E$, $X_2 = x_1E$, $X_3 = x_3^2E$, $X_4 = x_2E$, $X_5 = x_1x_3E$, $X_6 = x_3^3E$, $X_7 = x_1^2E$, $X_8 = x_2x_3E$, where $E$ is the Euler vector field

$$E = 3x_1\partial/\partial x_1 + 4x_2\partial/\partial x_2 + 2x_3\partial/\partial x_3. \tag{6.10}$$

**Proposition 6.21.** *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (U_8)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to $N$ is presented in Table 10.*

| $\mathcal{L}_{X_i}[\theta_j]$ | $[\theta_1]$ | $[\theta_2]$ | $[\theta_3]$ | $[\theta_4]$ | $[\theta_5]$ | $[\theta_6]$ | $[\theta_7]$ | $[\theta_8]$ |
|---|---|---|---|---|---|---|---|---|
| $X_0 = E$ | $5[\theta_1]$ | $6[\theta_2]$ | $7[\theta_3]$ | $7[\theta_4]$ | $8[\theta_5]$ | $9[\theta_6]$ | $10[\theta_7]$ | $11[\theta_8]$ |
| $X_1 = x_3E$ | $7[\theta_4]$ | $-16[\theta_5]$ | $3[\theta_6]$ | $9[\theta_6]$ | $10[\theta_7]$ | $11[\theta_8]$ | $[0]$ | $[0]$ |
| $X_2 = x_1E$ | $8[\theta_5]$ | $-6[\theta_6]$ | $-20[\theta_7]$ | $10[\theta_7]$ | $\frac{11}{3}[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_3 = x_3^2E$ | $9[\theta_6]$ | $-20[\theta_7]$ | $\frac{11}{3}[\theta_8]$ | $11[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_4 = x_2E$ | $-3[\theta_6]$ | $-40[\theta_7]$ | $-\frac{55}{3}[\theta_8]$ | $-\frac{11}{3}[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_5 = x_1x_3E$ | $10[\theta_7]$ | $-\frac{22}{3}[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_6 = x_3^3E$ | $11[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_7 = x_1^2E$ | $\frac{11}{3}[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_8 = x_2x_3E$ | $-\frac{11}{3}[\theta_8]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |

TABLE 10. Infinitesimal actions on algebraic restrictions of closed 2-forms to $U_8$. ($E$ is defined as in (6.10).)

Let $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + c_8\theta_8]_{U_8}$ be the algebraic restriction of a symplectic form $\omega$.

The first statement of Theorem 6.20 follows from the following lemmas.

**Lemma 6.22.** *If $c_1 \neq 0$ then the algebraic restriction $\mathcal{A} = [\sum_{k=1}^{8} c_k\theta_k]_{U_8}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{U_8}$.*

**Lemma 6.23.** *If $c_1 = 0$ and $c_2 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4]_{U_8}$.*

**Lemma 6.24.** *If $c_1 = c_2 = 0$ and $c_3 \neq 0$, $c_4 \neq 2c_3$, $c_4 \neq -\frac{1}{3}c_3$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{U_8}$.*

**Lemma 6.25.** *If $c_1 = c_2 = 0$ and $c_3 \neq 0$, $c_4 = -\frac{1}{3}c_3$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_3 - \frac{1}{3}\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{U_8}$.*

**Lemma 6.26.** *If $c_1 = c_2 = 0$ and $c_3 \neq 0$, $c_4 = 2c_3$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_3 + 2\theta_4 + \widetilde{c}_5\theta_5 + \widetilde{c}_7\theta_7]_{U_8}$.*

**Lemma 6.27.** *If $c_1 = c_2 = c_3 = 0$ and $c_4 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_4 + \widetilde{c}_5\theta_5]_{U_8}$.*

**Lemma 6.28.** *If $c_1 = 0, \dots, c_4 = 0$ and $c_5 \neq 0$, then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\pm\theta_5 + \widetilde{c}_6\theta_6]_{U_8}$.*

**Lemma 6.29.** *If $c_1 = 0, \dots, c_5 = 0$ and $c_6 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_6 + \widetilde{c}_7\theta_7]_{U_8}$.*

**Lemma 6.30.** *If $c_1 = 0, \dots, c_6 = 0$ and $c_7 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\pm\theta_7 + \widetilde{c}_8\theta_8]_{U_8}$.*

**Lemma 6.31.** *If $c_1 = 0, \dots, c_7 = 0$ and $c_8 \neq 0$ then the algebraic restriction $\mathcal{A}$ can be reduced by a symmetry of $U_8$ to an algebraic restriction $[\theta_8]_{U_8}$.*

The proofs of Lemmas 6.22 – 6.31 are similar and based on Table 8, Proposition 6.3 or the homotopy method.

To prove statement $(iii)$ of Theorem 6.20 we have to show that singularity classes corresponding to normal forms are disjoint. The singularity classes that can be distinguished by geometric conditions obviously are disjoint. From Theorem 4.3 we see that only classes $(U_8)^2$ and $(U_8)_5^{3,0}$ can not be distinguished by the geometric conditions but their symplectic multiplicities are distinct, hence the classes are disjoint.

The proofs of statements $(ii)$ and $(iv)$ of Theorem 6.20 are similar to analogous proofs for Theorem 6.9.

## 6.4. **Proofs for $U_9$ singularity.**

6.4.1. *Algebraic restrictions to $U_9$ and their classification.*

One has the following relations for ($U_9$)-singularities

$$[x_1^2 + x_2x_3]_{U_9} = 0. \tag{6.11}$$

$$[x_1x_2 + x_3^4]_{U_9} = 0, \tag{6.12}$$

$$[d(x_1^2 + x_2x_3)]_{U_9} = [2x_1dx_1 + x_2dx_3 + x_3dx_2]_{U_9} = 0 \tag{6.13}$$

$$[d(x_1x_2 + x_3^4)]_{U_9} = [x_1dx_2 + x_2dx_1 + 4x_3^3dx_3]_{U_9} = 0 \tag{6.14}$$

Multiplying these relations by suitable 1-forms and 2-forms we obtain the relations towards calculating $[\Lambda^2(\mathbb{R}^{2n})]_N$ for $N = U_9$.

**Proposition 6.32.** *The space $[\Lambda^2(\mathbb{R}^{2n})]_{U_9}$ is a 10-dimensional vector space spanned by the algebraic restrictions to $U_9$ of the 2-forms*

$\theta_1 = dx_1 \wedge dx_3$, $\theta_2 = dx_2 \wedge dx_3$, $\theta_3 = dx_1 \wedge dx_2$,

$\theta_4 = x_3dx_1 \wedge dx_3$, $\theta_5 = x_1dx_1 \wedge dx_3$, $\theta_6 = x_3^2dx_1 \wedge dx_3$, $\sigma = x_3dx_1 \wedge dx_2$,

$\theta_7 = x_1x_3dx_1 \wedge dx_3$, $\theta_8 = x_3^3dx_1 \wedge dx_3$, $\theta_9 = x_1x_3^2dx_1 \wedge dx_3$.

Proposition 6.32 and results of Section 6.1 imply the following description of the space $[Z^2(\mathbb{R}^{2n})]_{U_9}$ and the manifold $[\mathrm{Symp}(\mathbb{R}^{2n})]_{U_9}$.

**Theorem 6.33.** *The space $[Z^2(\mathbb{R}^{2n})]_{U_9}$ is a 9-dimensional vector space spanned by the algebraic restrictions to $U_9$ of the quasi-homogeneous 2-forms $\theta_i$ of degree $\delta_i$*

$\theta_1 = dx_1 \wedge dx_3$, $\delta_1 = 8$,

$\theta_2 = dx_2 \wedge dx_3$, $\delta_2 = 10$,

$\theta_3 = dx_1 \wedge dx_2$, $\delta_3 = 12$,

$\theta_4 = x_3dx_1 \wedge dx_3$, $\delta_4 = 11$,

$\theta_5 = x_1dx_1 \wedge dx_3$, $\delta_5 = 13$,

$\theta_6 = x_3^2dx_1 \wedge dx_3$, $\delta_6 = 14$,

$\theta_7 = x_1x_3dx_1 \wedge dx_3$, $\delta_7 = 16$,

$\theta_8 = x_3^3dx_1 \wedge dx_3$, $\delta_8 = 17$,

$\theta_9 = x_1x_3^2dx_1 \wedge dx_3$, $\delta_9 = 19$,

*If $n \geq 3$ then $[\mathrm{Symp}(\mathbb{R}^{2n})]_{U_9} = [Z^2(\mathbb{R}^{2n})]_{U_9}$. The manifold $[\mathrm{Symp}(\mathbb{R}^4)]_{U_9}$ is an open part of the 9-space $[Z^2(\mathbb{R}^4)]_{U_9}$ consisting of algebraic restrictions of the form $[c_1\theta_1 + \cdots + c_9\theta_9]_{U_9}$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$.*

**Theorem 6.34.**

*(i) Any algebraic restriction in $[Z^2(\mathbb{R}^{2n})]_{U_9}$ can be brought by a symmetry of $U_9$ to one of the normal forms $[U_9]^i$ given in the second column of Table 11.*

*(ii) The codimension in $[Z^2(\mathbb{R}^{2n})]_{U_9}$ of the singularity class corresponding to the normal form $[U_9]^i$ is equal to $i$, the symplectic multiplicity and the index of isotropy are given in the fourth and fifth columns of Table 11.*

*(iii) The singularity classes corresponding to the normal forms are disjoint.*

*(iv) The parameters $c, c_1, c_2, c_3$ of the normal forms $[U_9]^i$ are moduli.*

The proof of Theorem 6.34 is presented in Section 6.4.3.

| symplectic class | normal forms for algebraic restrictions | cod | $\mu^{\mathrm{sym}}$ | ind |
|---|---|---|---|---|
| $(U_9)^0$ $(2n \geq 4)$ | $[U_9]^0 : [\pm\theta_1 + c_1\theta_2 + c_2\theta_3]_{U_9}$, | 0 | 2 | 0 |
| $(U_9)^1$ $(2n \geq 4)$ | $[U_9]^1 : [\pm\theta_2 + c_1\theta_3 + c_2\theta_4 + c_3\theta_6]_{U_9}$ | 1 | 4 | 0 |
| $(U_9)^2$ $(2n \geq 4)$ | $[U_9]^2 : [\pm\theta_3 + c_1\theta_4 + c_2\theta_5]_{U_9}$, $c_1 \neq 0$ | 2 | 4 | 0 |
| $(U_9)^{3,0}$ $(2n \geq 4)$ | $[U_9]^{3,0} : [\pm\theta_3 + c_1\theta_5 + c_2\theta_6]_{U_9}$, $c_1 \neq 0$ | 3 | 5 | 0 |
| $(U_9)^{4,0}$ $(2n \geq 4)$ | $[U_9]^{4,0} : [\pm\theta_3 + c_1\theta_6 + c_2\theta_7]_{U_9}$ | 4 | 6 | 0 |
| $(U_9)^{3,1}$ $(2n \geq 6)$ | $[U_9]^{3,1} : [\theta_4 + c\theta_5]_{U_9}$ | 3 | 4 | 1 |
| $(U_9)^{4,1}$ $(2n \geq 6)$ | $[U_9]^{4,1} : [\theta_5 + c_1\theta_6 + c_2\theta_8]_{U_9}$ | 4 | 6 | 1 |
| $(U_9)^5$ $(2n \geq 6)$ | $[U_9]^5 : [\pm\theta_6 + c\theta_7]_{U_9}$ | 5 | 6 | 2 |
| $(U_9)^6$ $(2n \geq 6)$ | $[U_9]^6 : [\pm\theta_7 + c\theta_8]_{U_9}$ | 6 | 7 | 2 |
| $(U_9)^7$ $(2n \geq 6)$ | $[U_9]^7 : [\theta_8 + c\theta_9]_{U_9}$ | 7 | 8 | 3 |
| $(U_9)^8$ $(2n \geq 6)$ | $[U_9]^8 : [\theta_9]_{U_9}$ | 8 | 8 | 3 |
| $(U_9)^9$ $(2n \geq 6)$ | $[U_9]^9 : [0]_{U_9}$ | 9 | 9 | $\infty$ |

TABLE 11. Classification of symplectic $U_9$ singularities.
*cod* – codimension of the classes; $\mu^{sym}$– symplectic multiplicity;
*ind* – the index of isotropy.

6.4.2. *Symplectic normal forms.*

Let us transfer the normal forms $[U_9]^i$ to symplectic normal forms. We fix a family $\omega^i$ of symplectic forms on $\mathbb{R}^{2n}$ realizing the family $[U_9]^i$ of algebraic restrictions.

$\omega^0 = \pm\theta_1 + c_1\theta_2 + c_2\theta_3 + dx_2 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^1 = \pm\theta_2 + c_1\theta_3 + c_2\theta_4 + c_3\theta_6 + dx_1 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^2 = \pm\theta_3 + c_1\theta_4 + c_2\theta_5 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n},\ c_1 \neq 0$;
$\omega^{3,0} = \pm\theta_3 + c_1\theta_5 + c_2\theta_6 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n},\ c_1 \neq 0$;
$\omega^{4,0} = \pm\theta_3 + c_1\theta_6 + c_2\theta_7 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^{3,1} = \theta_4 + c\theta_5 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^{4,1} = \theta_5 + c_1\theta_6 + c_2\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^5 = \pm\theta_6 + c\theta_7 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^6 = \pm\theta_7 + c\theta_8 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^7 = \theta_8 + c\theta_9 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^8 = \theta_9 + dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$;
$\omega^9 = dx_1 \wedge dx_4 + dx_2 \wedge dx_5 + dx_3 \wedge dx_6 + dx_7 \wedge dx_8 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

6.4.3. *Proof of Theorem 6.34.* The germ of a vector field tangent to $U_9$ of non trivial action on algebraic restrictions of closed 2-forms to $U_9$ may be described as a linear combination of germs of vector fields: $X_0 = E, X_1 = x_3E, X_2 = x_1E, X_3 = x_3^2E, X_4 = x_2E, X_5 = x_1x_3E,\ X_6 = x_3^3E, X_7 = x_1x_3^2E$, where $E$ is the Euler vector field

$$E = 5x_1\partial/\partial x_1 + 7x_2\partial/\partial x_2 + 3x_3\partial/\partial x_3. \tag{6.15}$$

**Proposition 6.35.** *The infinitesimal action of germs of quasi-homogeneous vector fields tangent to $N \in (U_9)$ on the basis of the vector space of algebraic restrictions of closed 2-forms to $N$ is presented in Table 12.*

| $\mathcal{L}_{X_i}[\theta_j]$ | $[\theta_1]$ | $[\theta_2]$ | $[\theta_3]$ | $[\theta_4]$ | $[\theta_5]$ | $[\theta_6]$ | $[\theta_7]$ | $[\theta_8]$ | $[\theta_9]$ |
|---|---|---|---|---|---|---|---|---|---|
| $X_0=E$ | $8[\theta_1]$ | $10[\theta_2]$ | $12[\theta_3]$ | $11[\theta_4]$ | $13[\theta_5]$ | $14[\theta_6]$ | $16[\theta_7]$ | $17[\theta_8]$ | $19[\theta_9]$ |
| $X_1=x_3E$ | $11[\theta_4]$ | $-26[\theta_5]$ | $[0]$ | $14[\theta_6]$ | $16[\theta_7]$ | $17[\theta_8]$ | $19[\theta_9]$ | $[0]$ | $[0]$ |
| $X_2=x_1E$ | $13[\theta_5]$ | $[0]$ | $-68[\theta_8]$ | $16[\theta_7]$ | $[0]$ | $19[\theta_9]$ | $0[0]$ | $[0]$ | $[0]$ |
| $X_3=x_3^2E$ | $14[\theta_6]$ | $-32[\theta_7]$ | $[0]$ | $17[\theta_8]$ | $19[\theta_9]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_4=x_2E$ | $[0]$ | $-136[\theta_8]$ | $-38[\theta_9]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_5=x_1x_3E$ | $16[\theta_7]$ | $[0]$ | $[0]$ | $19[\theta_9]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_6=x_3^3E$ | $17[\theta_8]$ | $-38[\theta_9]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |
| $X_7=x_1x_3^2E$ | $19[\theta_9]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ | $[0]$ |

TABLE 12. Infinitesimal actions on algebraic restrictions of closed 2-forms to $U_9$. ($E$ is defined as in (6.15).)

Let $\mathcal{A} = [c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + c_4\theta_4 + c_5\theta_5 + c_6\theta_6 + c_7\theta_7 + c_8\theta_8 + c_9\theta_9]_{U_9}$ be the algebraic restriction of a symplectic form $\omega$.

The first statement of Theorem 6.34 follows from the following lemmas.

**Lemma 6.36.** *If* $c_1 \neq 0$ *then the algebraic restriction* $\mathcal{A} = [\sum_{k=1}^{9} c_k\theta_k]_{U_9}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_1 + \widetilde{c}_2\theta_2 + \widetilde{c}_3\theta_3]_{U_9}$.

**Lemma 6.37.** *If* $c_1=0$ *and* $c_2 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_2 + \widetilde{c}_3\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_6\theta_6]_{U_9}$.

**Lemma 6.38.** *If* $c_1 = c_2 = 0$ *and* $c_3 \cdot c_4 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_3 + \widetilde{c}_4\theta_4 + \widetilde{c}_5\theta_5]_{U_9}$.

**Lemma 6.39.** *If* $c_1 = c_2 = c_4 = 0$ *and* $c_3 \cdot c_5 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_3 + \widetilde{c}_5\theta_5 + \widetilde{c}_6\theta_6]_{U_9}$.

**Lemma 6.40.** *If* $c_1 = c_2 = c_4 = c_5 = 0$ *and* $c_3 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_3 + \widetilde{c}_6\theta_6 + \widetilde{c}_7\theta_7]_{U_9}$.

**Lemma 6.41.** *If* $c_1 = c_2 = c_3 = 0$ *and* $c_4 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\theta_4 + \widetilde{c}_5\theta_5]_{U_9}$.

**Lemma 6.42.** *If* $c_1 = 0, \ldots, c_4 = 0$ *and* $c_5 \neq 0$, *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\theta_5 + \widetilde{c}_6\theta_6 + \widetilde{c}_8\theta_8]_{U_9}$.

**Lemma 6.43.** *If* $c_1 = 0, \ldots, c_5 = 0$ *and* $c_6 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_6 + \widetilde{c}_7\theta_7]_{U_9}$.

**Lemma 6.44.** *If* $c_1 = 0, \ldots, c_6 = 0$ *and* $c_7 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\pm\theta_7 + \widetilde{c}_8\theta_8]_{U_9}$.

**Lemma 6.45.** *If* $c_1 = 0, \ldots, c_7 = 0$ *and* $c_8 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\theta_8 + \widetilde{c}_9\theta_9]_{U_9}$.

**Lemma 6.46.** *If* $c_1 = 0, \ldots, c_8 = 0$ *and* $c_9 \neq 0$ *then the algebraic restriction* $\mathcal{A}$ *can be reduced by a symmetry of* $U_9$ *to an algebraic restriction* $[\theta_9]_{U_9}$.

The proofs of Lemmas 6.36 – 6.46 are similar and based on Table 12, Proposition 6.3 or the homotopy method.

The proofs of statements $(ii)$ – $(iv)$ of Theorem 6.34 are similar to analogous proofs for Theorem 6.9.

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Warsaw University of Technology
Faculty of Mathematics and Information Science
Koszykowa 75, 00-662 Warszawa, Poland
*E-mail address*: ztrebska@mini.pw.edu.pl